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Migration with the full acoustic wave equation

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ABSTRACT

Conventional finite-difference migration has relied on one-way wave equations which allow energy to propagate only downward. Although generally reliable, such equations may not give accurate migration when the structures have strong lateral velocity variations or steep dips. The present study examined an alternative approach based on the full acoustic wave equation. The migration algorithm which developed from this equation was tested against synthetic data and against physical model data. The results indicated that such a scheme gives accurate migration for complicated structures.

INTRODUCTION

During the last decade, finite-difference migration has been dominated by the approach of Claerbout (1972). His method uses one-way wave equations which allow energy to propagate only downward. Although successful in many situations, the method is limited by the assumptions made in deriving the one-way wave equations. In particular, it is assumed that spatial derivatives of the velocity field can be ignored (Claerbout, 1972, 1976; Stolt, 1978; Gazdag, 1980; Berkhout and Palthe, 1979). However, such terms are significant in the presence of strong velocity contrasts. Furthermore, most finite-difference migration schemes which use the one-way wave equation contain a limit on the maximum dip of events which can be migrated properly. One-way wave equations are also incapable by nature of producing correct amplitudes (Berkhout and Palthe, 1979; Larner et al, 1981). The change in amplitude for smooth and discontinuous velocity variations is discussed in the Appendix for the one-dimensional case. This becomes important in the construction of before stack migration methods.

The present study tackled the problem of finding a migration scheme that would eliminate the deficiencies outlined above, offering an improvement over conventional finite-difference migration algorithms. A migration scheme for stacked sections based on the full acoustic wave equation is introduced. Because such an equation is directly derivable from continuum mechanics, it seemed likely to give accurate migration in areas with high velocity contrasts and steep dips. The main limitations which would remain in the scheme were those associated with the conceptual model on which migration is based. For migration of stacked or zero-offset sections, this model is the so-called "exploding reflector model" (Loewenthal et al, 1976) which cannot correctly account for multiples or for complicated wave propagation such as, for example, a diffraction followed by a reflection.

Our study found that in order to implement migration based on the full acoustic wave equation, a number of obstacles had to be overcome. First, the acoustic wave equation is of second order in the spatial coordinates, and therefore requires two boundary conditions to initiate the depth extrapolation. On the other hand, the seismic data contain only one recorded field which is related either to the pressure or to the vertical pressure gradients. Second, a means must be supplied for eliminating evanescent energy which, if not removed, can cause numerical solutions to grow exponentially out of bounds. These topics are dealt with in later sections.

In the following sections we describe the numerical solution method developed for the migration algorithm. The scheme is then tested against simple synthetic examples, and against physical modeling data collected in the modeling tank at the Seismic Acoustics Laboratory at the Univ. of Houston.

BASIC EQUATIONS

The migration algorithm is derived in the space-frequency domain, with the acoustic wave equation serving as the basis. In an acoustic medium with variable density and velocity, the acoustic wave equation reads

$$\frac{\partial}{\partial x}\left(\frac{1}{\rho}\frac{\partial P}{\partial x}\right)+\frac{\partial}{\partial z}\left(\frac{1}{\rho}\frac{\partial P}{\partial z}\right)=\frac{1}{C^2\rho}\frac{\partial^2 P}{\partial t^2}, \qquad (1)$$

where x and z are the horizontal and vertical coordinates, $P(x, z, t)$ is the pressure field at time t, $C(x, z)$ is the acoustic velocity, and $\rho(x, z)$ is the density.

For the migration algorithm, equation (1) is Fourier transformed with respect to time and rewritten as a set of two coupled equations:

$$\frac{\partial}{\partial z}\begin{bmatrix}\tilde{P}\\ \frac{1}{\rho}\frac{\partial\tilde{P}}{\partial z}\end{bmatrix}=\begin{bmatrix}0 & \rho\\ \frac{-\omega^2}{\rho C^2}-\frac{\partial}{\partial x}\left(\frac{1}{\rho}\frac{\partial}{\partial x}\right) & 0\end{bmatrix}\begin{bmatrix}\tilde{P}\\ \frac{1}{\rho}\frac{\partial\tilde{P}}{\partial z}\end{bmatrix}, \qquad (2)$$

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where ω is the temporal frequency, and $\tilde{P}$ and $\partial \tilde{P}/\partial z$ represent the transformed pressure and vertical pressure gradient, respectively.

Equation (2) is written consistently with the continuity conditions of continuum mechanics which require that both the tractions and the displacements remain continuous across all possible interfaces in the medium. This follows because in an acoustic medium the tractions are equal to the pressures, and the accelerations are equal to $(1/\rho)(\partial P/\partial x)$ and $(1/\rho)(\partial P/\partial z)$, respectively.

When the density is assumed constant throughout the medium, equation (2) simplifies to give

$$\frac{\partial}{\partial z}\begin{bmatrix} \tilde{P} \\ \\ \dfrac{\partial \tilde{P}}{\partial z} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ \\ \dfrac{-\omega^2}{C^2} - \dfrac{\partial^2}{\partial x^2} & 0 \end{bmatrix} \begin{bmatrix} \tilde{P} \\ \\ \dfrac{\partial \tilde{P}}{\partial z} \end{bmatrix}. \tag{3}$$

In the following, equation (3) will be used for simplicity. It is useful in practice since the density usually varies less than the velocity in geologic structures, and is often not known accurately. However, the constant density assumption is not fundamental to the present migration scheme.

The depth extrapolation in the migration is based on equation (3). For stacked or zero-offset time sections, x and z represent midpoint coordinates and $C(x, z)$ is equal to half the true acoustic velocity in the medium (for further discussion, see Loewenthal et al, 1976; Stolt, 1978). The input section for the migration is given by $P(x, z = 0, t)$ and the migrated section is given by $P(x, z, t = 0)$ (Loewenthal et al, 1976). The integration of equation (3) is carried out for each frequency component separately. At each depth level z, components of the vector

$$\begin{bmatrix} \tilde{P}\ (x, z, \omega) \\ \dfrac{\partial \tilde{P}}{\partial z}(x, z, \omega) \end{bmatrix}$$

are stepped down to the next depth level $z + \Delta z$ for all x. The process is initiated at the surface after a specification of $\tilde{P}(x, 0, \omega)$ and $(\partial \tilde{P}/\partial z)(x, 0, \omega)$. The final migrated section $P(x, z, 0)$ is obtained after $\tilde{P}(x, z, \omega)$ has been calculated for all ω by summing over all ω (Gazdag, 1980):

$$P(x, z, 0) = \sum_{\omega} \tilde{P}(x, z, \omega). \tag{4}$$

METHOD OF EXTRAPOLATION IN DEPTH

For the depth extrapolation, equation (3) is spatially discretized. Let N_x be the number of seismic traces and Δx the trace spacing. We denote by $\tilde{P}(i, z, \omega)$ and $(\partial \tilde{P}/\partial z)(i, z, \omega)$ the respective values of the transformed pressure and transformed vertical pressure gradient at depth z, and at horizontal location $x = x_0 + (i - 1)\Delta x$. With this discretization in x, and with an appropriate approximation to $\partial^2 P/\partial x^2$, equation (3) becomes a set of $2N_x$ coupled ordinary differential equations in z for $P(i, z, \omega)$ and $(\partial \tilde{P}/\partial z)(i, z, \omega)$, $i = 1, \ldots, N_x$. The integration in depth can then be carried out with standard solution techniques for ordinary differential equations. We used a fourth-order Runge-Kutta method because it is accurate and easy to implement on an array processor.

Using the discretized version of equation (3) requires an approximation for $\partial^2 \tilde{P}/\partial x^2$. The present scheme uses a Fourier method to calculate this term (Kosloff and Baysal, 1982; Gazdag, 1980). Accordingly, the pressure transforms $\tilde{P}(i, z, \omega)$ are spatially transformed by a fast Fourier transform (FFT) to yield $\tilde{\tilde{P}}(K_x, z, \omega)$, then multiplied by $-K_x^2$ and inverse transformed to give $(\partial^2 \tilde{P}/\partial x^2)(i, z, \omega)$. This derivative operator, unlike finite-differences, is accurate to the spatial Nyquist frequency. It also allows easy removal of undesirable spatial wavenumber components such as evanescent waves.

GENERATION OF SURFACE VALUES OF $\tilde{P}$ and $\partial \tilde{P}/\partial z$

The initiation of migration based on equation (3) requires the specification of both $\tilde{P}$ and $\partial \tilde{P}/\partial z$ on the surface. Since only one of these fields is recorded in practice, the remaining field must be generated from mathematical assumptions.

We assume that the seismic data are given by the pressure field $P(x, 0, t)$. $\tilde{P}(x, 0, \omega)$ can then be calculated from $P(x, 0, t)$ by Fourier transformation in time. The values of $(\partial \tilde{P}/\partial z)(x, 0, \omega)$ need to be generated from $\tilde{P}(x, 0, \omega)$ mathematically (other cases in which the recorded field is not the pressure can be treated by the same method outlined in this section). For this process we assume that the acoustic velocity is laterally uniform in the vicinity of the surface (but can be completely nonuniform everywhere else), and that the seismic time section consists of upgoing energy only. The latter assumption is also used in migration schemes based on one-way equations (Claerbout, 1976).

In a region in which the acoustic velocity C is constant, equation (1) can be doubly transformed in x and t to give

$$\frac{\partial^2 \tilde{\tilde{P}}}{\partial z^2} = -\left(\frac{\omega^2}{C^2} - K_x^2\right) \tilde{\tilde{P}}(K_x, z, \omega), \tag{5}$$

where $\tilde{\tilde{P}}$ is the twice transformed pressure and K_x is the horizontal wavenumber. The solutions to equation (5) are given by

$$\tilde{\tilde{P}}(K_x, z, \omega) = e^{i\eta z}\tilde{\tilde{P}}(K_x, 0, \omega), \tag{6}$$

with $\eta = \sqrt{\omega^2/C^2 - K_x^2}$. The solution (6) includes only upgoing waves under the convention that z increases with depth. This study uses only nonevanescent energy components for which $\omega^2/C^2 > K_x^2$.

The doubly transformed pressure gradients $(\partial \tilde{\tilde{P}}/\partial z)(K_x, z, \omega)$ can be obtained from equation (6) by differentiation,

$$\frac{\partial \tilde{\tilde{P}}}{\partial z} = i\eta\, e^{i\eta z}\tilde{\tilde{P}}(K_x, 0, \omega) = i\eta\, \tilde{\tilde{P}}(K_x, z, \omega). \tag{7}$$

The generated vertical pressure gradients $(\partial \tilde{P}/\partial z)(x, 0, \omega)$ on the surface are obtained from equation (7) by setting $z = 0$ and by an inverse Fourier transformation with respect to x.

The procedure for generation of $\partial \tilde{P}/\partial z$ on the surface is compatible with assumptions used in most migration schemes. However, other alternatives may fit the reality of the field configuration better. For example, it may be more appropriate for data from land surveys to set the pressures $P(x, 0, t)$ equal to zero and to assume that the recorded data are proportional to $(\partial P/\partial z)$ $(x, 0, t)$. However, we did not attempt to pursue these alternatives in the present study.

ELIMINATION OF EVANESCENT ENERGY

Evanescent waves are given by the exponentially varying solutions of the wave equation. Although in physical reality only solutions which exponentially decrease with depth are present, in numerical algorithms the exponentially increasing solutions can also be generated and cause the numerical results to grow out

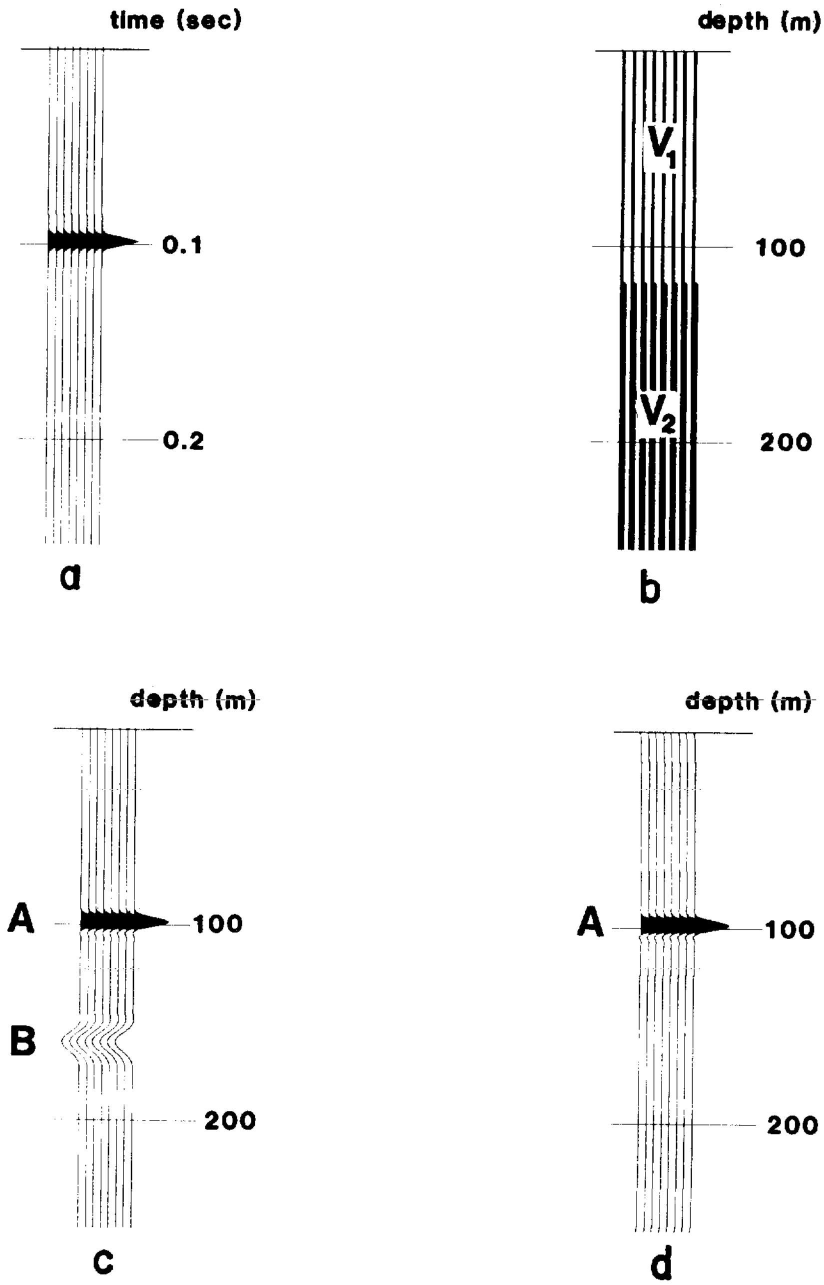

FIG. 1. (a) A one-dimensional time section containing a single event. (b) Velocity model $V_1 = 2000$ m/sec, $V_2 = 1000$ m/sec. (c) Migrated depth section based on equation (3). (d) Migrated section after the elimination of downgoing energy at time $t = 0$.

of bounds. Therefore, it is important to eliminate evanescent energy in implementing migration with the full acoustic wave equation.

For a laterally uniform medium with a depth dependent velocity $C(z)$, the evanescent solutions are defined by the relation

$$K_x > \frac{\omega}{C(z)},$$

where K_x is the horizontal wavenumber and ω is the temporal frequency. When the velocity field varies laterally, the identification of the evanescent field becomes less clear cut. In this work we chose the definition

$$K_x > \frac{\omega}{C_{\max}}, \tag{8}$$

where $C_{\max}$ is the highest velocity at the depth z. Our experience indicates that criterion (8) assures numerical stability. However, in some cases it may cause the elimination of steeply dipping events in low-velocity regions. In applications, a less stringent condition than inequality (8) may sometimes be preferable.

ELIMINATION OF DOWNGOING ENERGY AT TIME ZERO

The conceptual basis for the migration of zero-offset data is the exploding reflector model (Loewenthal et al, 1976). According to this model, a zero-offset time section can be generated directly by halving all the acoustic velocities in the medium and placing explosive sources on the reflecting horizons. These explode at time zero with strengths proportional to the reflection coefficients. The need to replace physical reality by a model stems from the fact that a zero-offset section cannot be obtained from a single shot, but rather is composed from a series of shots. With the exploding reflector model, the aim of migration is to produce the pressure at time zero in all space (Loewenthal et al, 1976; Stolt, 1978).

The exploding reflector model applies directly to migration based on one-way wave equations since these propagate only upgoing energy. For the two-way wave equation, there is a nonuniqueness concerning downgoing energy. Consider the one-dimensional (1-D) time section in Figure 1a. When the velocity

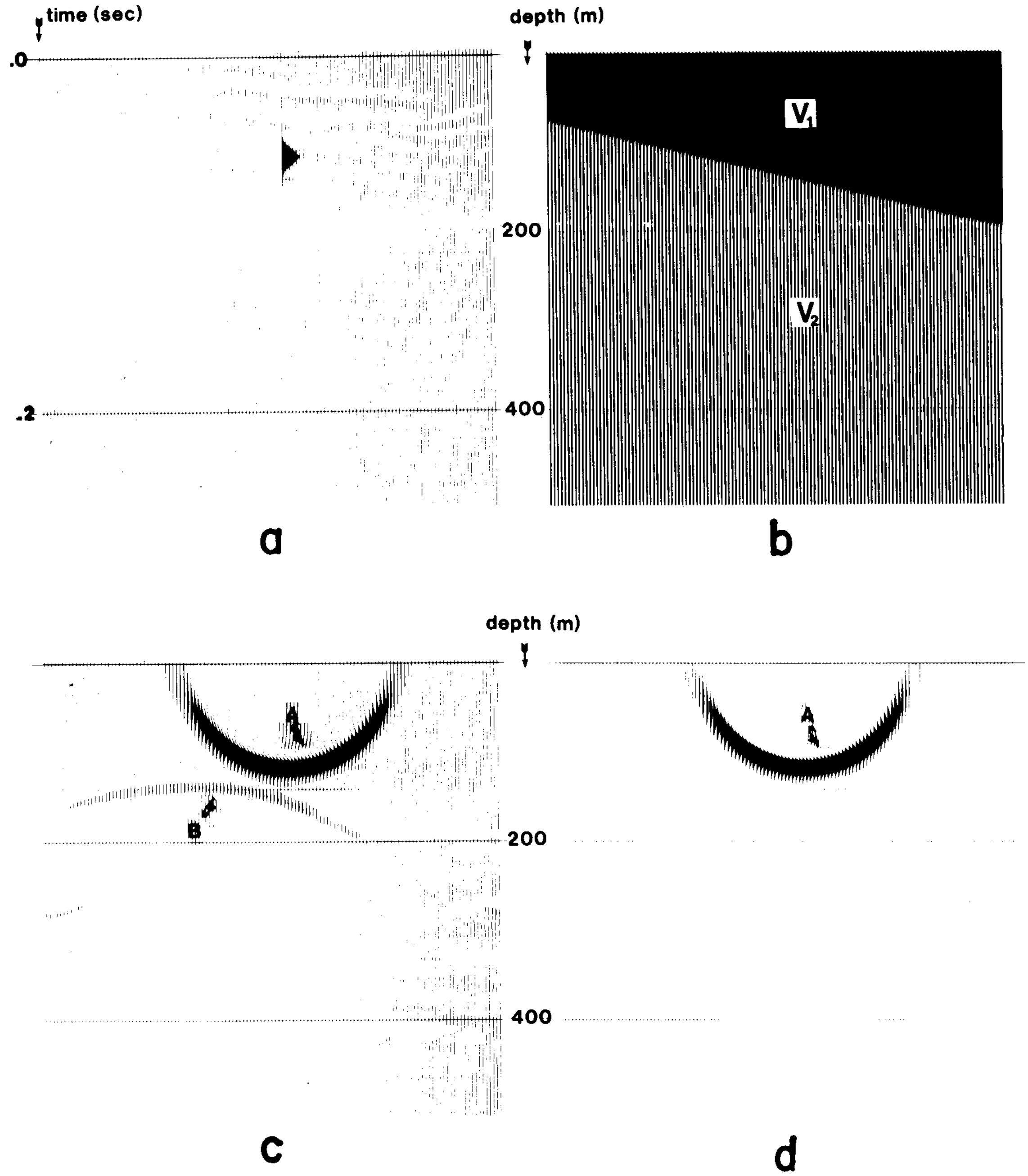

FIG. 2. (a) A time section containing a single spike. (b) Velocity model $V_1 = 2000$ m/sec, $V_2 = 1000$ m/sec. (c) Migrated section based on equation (3). (d) Migrated section after the elimination of downgoing energy at time $t = 0$.

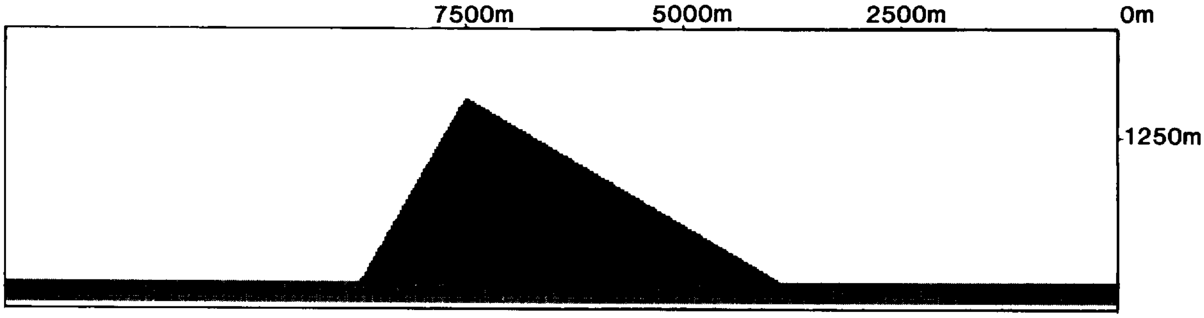

FIG. 3. Spatial configuration of the wedge model.

structure is as shown in Figure 1b, the migrated section from the solution of equation (3) becomes the section shown in Figure 1c. This section contains an upgoing event A which corresponds to the single event on the time section in Figure 1a. An additional event B containing downgoing energy only is also present. This event appears because of an inherent nonuniqueness in the conceptual model on which the migration is based. With the model, a surface recording alone cannot determine the amount of downgoing energy at time $t = 0$. In order to determine the amount of downgoing energy, a set of geophones also needs to be placed beneath the structure of interest. Obviously this option is not realizable in practice.

Downgoing energy at time $t = 0$ can be eliminated from the depth section by filtering out components with negative vertical wavenumbers. When this procedure is applied to the section of Figure 1c, the section shown in Figure 1d results. This section contains event A only.

Figure 2 shows the same elimination method for a two-dimensional (2-D) example. The time section in Figure 2a contains a single event which should give a depth section with a circular reflector (event A in Figure 2c). However, when the velocity is as in Figure 2b, an additional downgoing event B is produced on the depth section (Figure 2c). After the elimination procedure is applied, only event A remains (Figure 2d).

The nonuniqueness associated with downgoing energy becomes significant only in the presence of strong velocity contrasts. In many cases elimination of this energy is not necessary.

RELATION OF MIGRATION SCHEME TO THE PHASE-SHIFT METHOD

In a uniform or horizontally stratified region the migration scheme of this study is closely related to the phase-shift method (Gazdag, 1978). The point of departure is that the depth extrapolation is done here numerically instead of by phase shift.

For a homogeneous region with acoustic velocity C, equation (3) can be transformed with respect to x to give

$$\frac{\partial}{\partial z}\begin{bmatrix} \tilde{P} \\ \\ \dfrac{\partial \tilde{P}}{\partial z} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ \\ -\eta^2 & 0 \end{bmatrix}\begin{bmatrix} \tilde{P} \\ \\ \dfrac{\partial \tilde{P}}{\partial z} \end{bmatrix}, \tag{9}$$

where $\eta^2 = \omega^2/C^2 - K_x^2$, and $\tilde{P}(K_x, z, \omega)$ and $(\partial\tilde{P}/\partial z)(K_x, z, \omega)$ are, respectively, the doubly transformed pressure and vertical pressure gradient.

The solutions of equation (9) are given by

$$\tilde{P}(K_x, z, \omega) = A_+ e^{i\eta(z-z_0)} + A_- e^{-i\eta(z-z_0)}], \tag{10}$$

and

$$\frac{\partial \tilde{P}}{\partial z} = i\eta[A_+ e^{i\eta(z-z_0)} - A_- e^{-i\eta(z-z_0)}], \tag{11}$$

where A_+ and A_-, respectively, represent amplitudes of upgoing and downgoing waves and z_0 is a reference depth. Equations (10) and (11) can be used for depth extrapolation in a phase-shift migration. This migration can also be used for a horizontally stratified velocity model by using equations (10) and (11) within each layer and determining A_+ and A_- by the continuity conditions of $\tilde{P}$ and $\partial\tilde{P}/\partial z$ across the top interface of the layer. This type of phase-shift migration accounts for amplitude changes. When A_- is set equal to zero, only upgoing energy is considered and $\partial\tilde{P}/\partial z$ can no longer be made continuous. The migration then becomes the phase-shift method in Gazdag (1978). In this migration A_+ remains constant at all depths and the correct amplitudes of events are no longer restored (Appendix). The migration is then equivalent to migration with one-way wave equations.

When the migration method of this paper is compared with the phase-shift method for a stratified medium, it becomes apparent that the two methods are identical in the manner in which horizontal derivatives are calculated [the Fourier method calculates $\partial^2 P/\partial x^2$ in the (K_x, z, ω) domain by multiplication by $-K_x^2$]. The only difference between the two methods is that in this study, $\tilde{P}$ and $\partial\tilde{P}/\partial z$ are stepped down by a Runge-Kutta method, instead of by equations (10) and (11). Therefore, for sufficiently small Δz the two methods should give practically identical results.

EXAMPLE: A BURIED WEDGE STRUCTURE

In the example of a buried wedge structure, the input time section was obtained from the acoustic modeling tank at the Seismic Acoustics Laboratory at the Univ. of Houston. The same model was used in Kosloff and Baysal (1982) to compare forward modeling results. The scaled dimensions of the model are shown in Figure 3. The wedge structure was made of low-velocity room temperature vulcanized (RTV) rubber, whereas the base was made of high-velocity Plexiglas. The whole model was immersed in water which had a scaled velocity of 3950 m/sec. A zero-offset

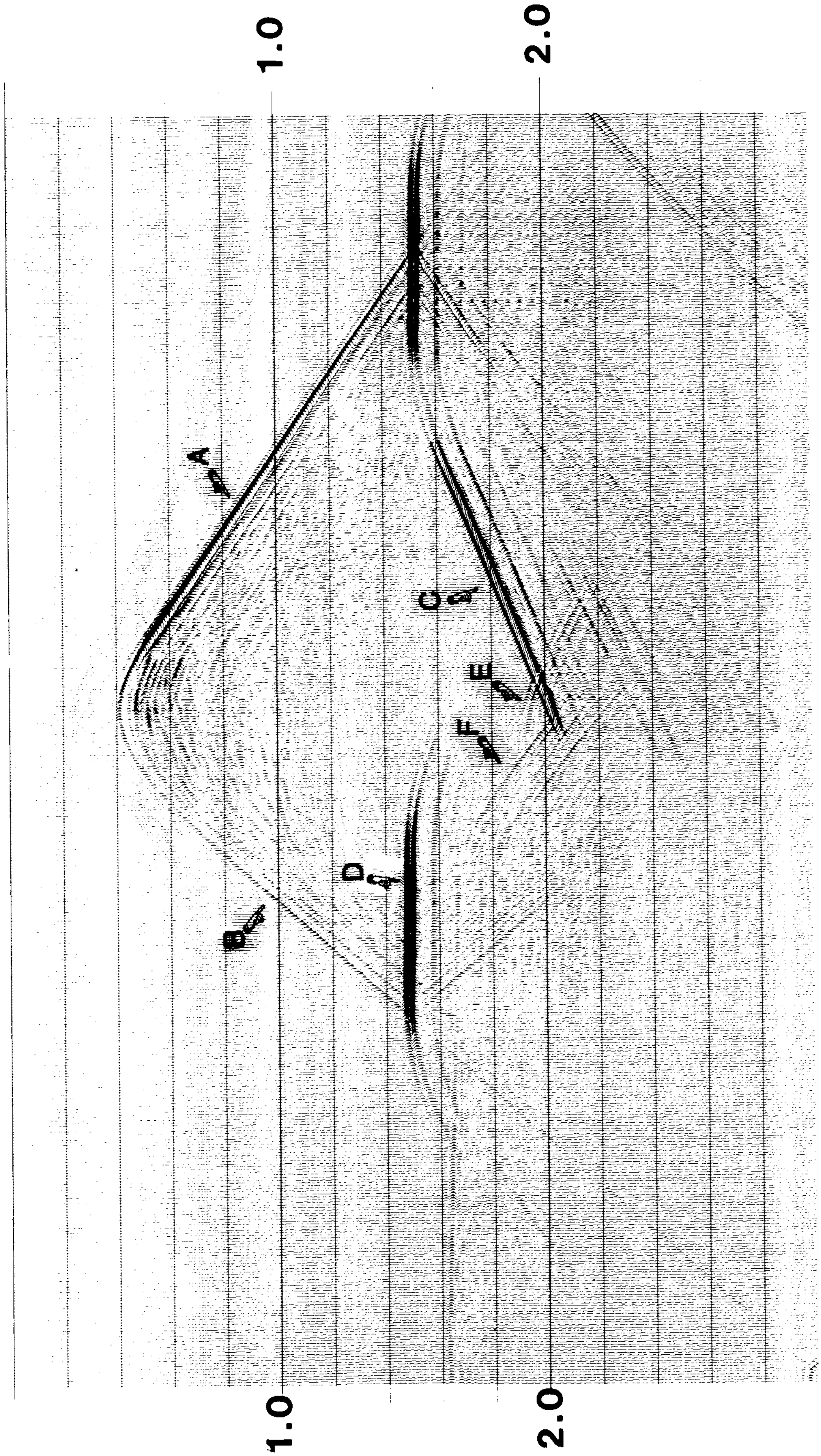

FIG. 4. Time section from a zero-offset line shot in the physical modeling tank.

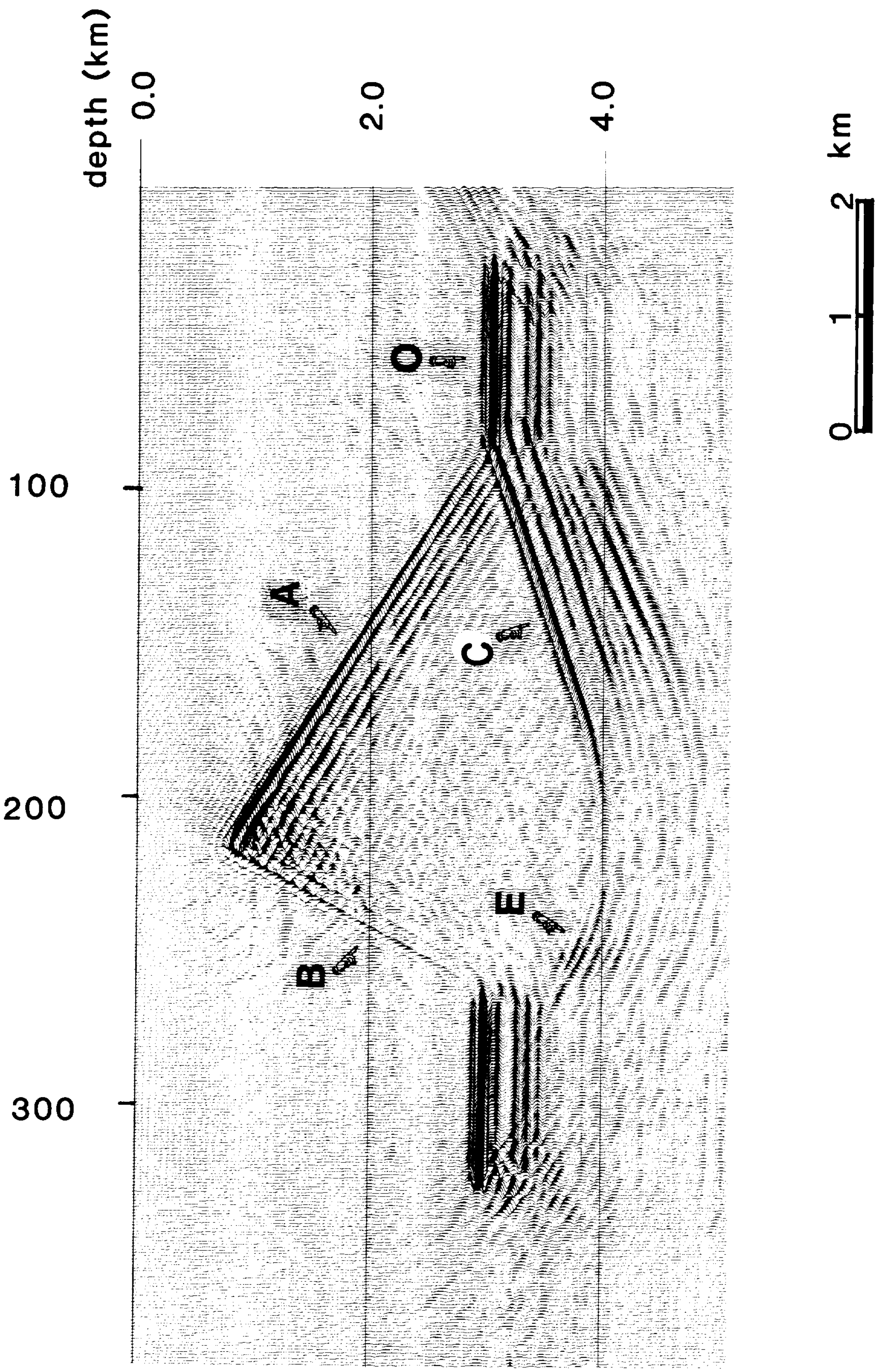

FIG. 5. Migrated section with uniform velocity of 3950 m/sec.

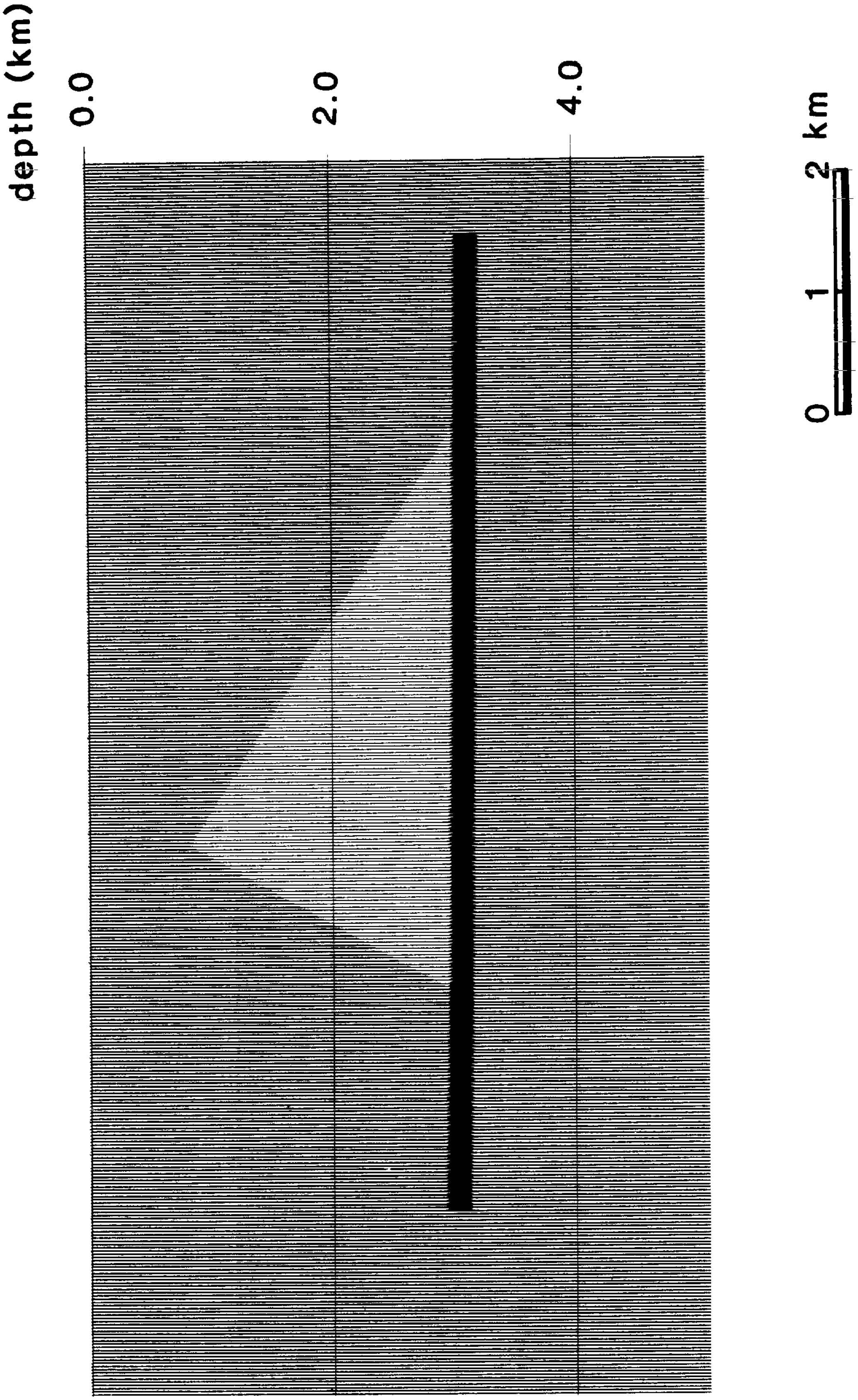

FIG. 6. Velocity structure for variable velocity migration.

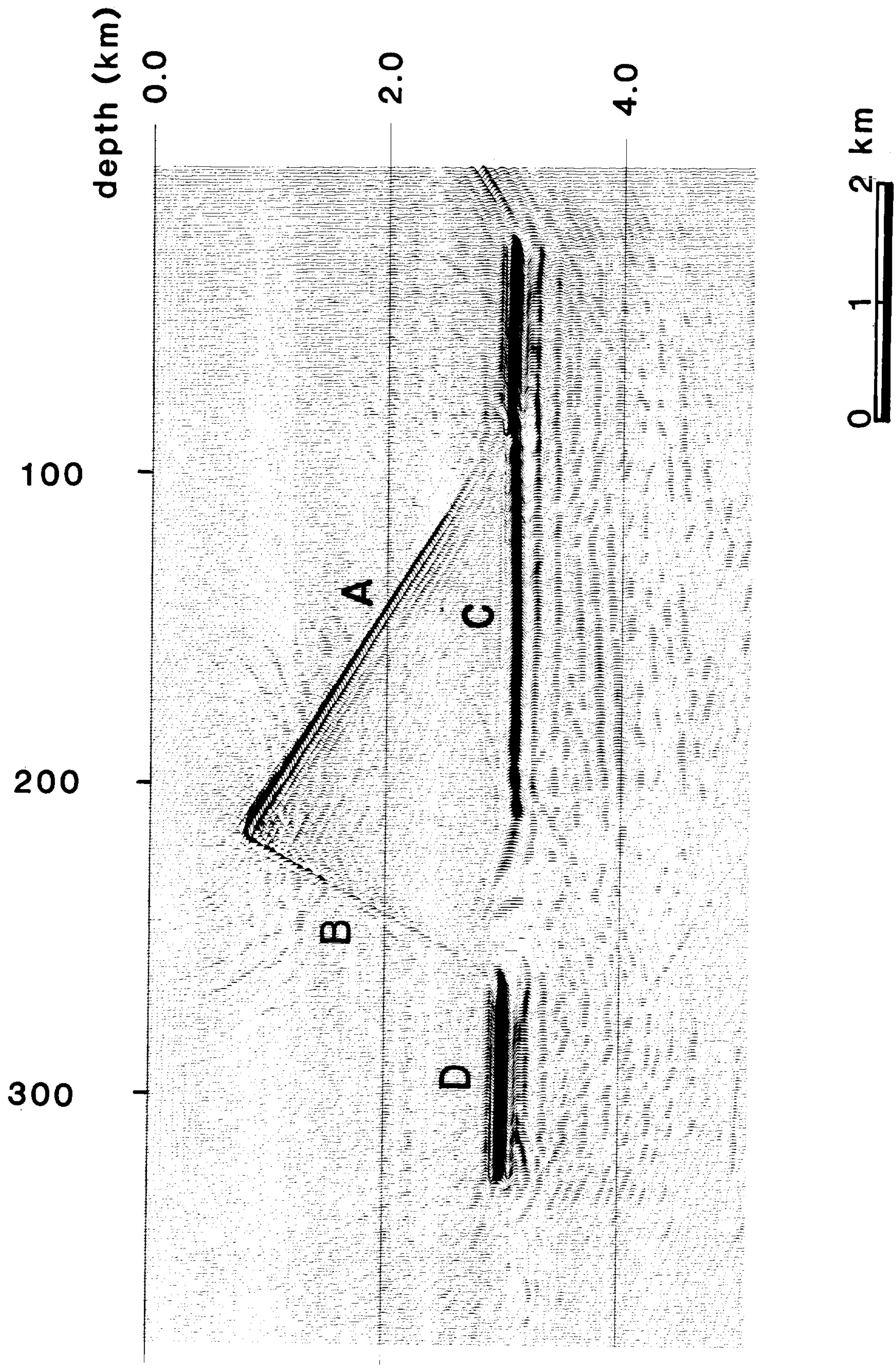

FIG. 7. Migrated depth section with variable velocity.

line was collected perpendicular to the symmetry axis of the wedge at a scaled height of approximately 800 m above the wedge tip, with a shot spacing of 26 m. Since this model includes steep dips and high velocity contrasts, it serves as a good test for the migration algorithm.

Figure 4 shows the observed time section. Events A and B are interpreted as reflections from the sloping sides of the wedge, whereas events C and D are reflections from the Plexiglas base. Events E and F are complicated events not accounted for by the exploding reflector model (Kosloff and Baysal, 1982).

In order to make the migration as objective as possible, no prior knowledge of the structure was assumed and at first the time section (Figure 4) was migrated with a uniform velocity equal to the scaled water velocity of 3950 m/sec. The migrated section is shown in Figure 5. In this figure events which traveled to the surface through water only were migrated to their respective proper positions including the 60 degree wedge interface B. On the other hand, the reflection C from the Plexiglas base underlying the wedge is undermigrated.

In the second stage, variable velocities were introduced. The velocity interfaces, except portions of the base under the wedge, were derived from the depth section of the constant velocity migration (Figure 5). The portion of the base under the RTV wedge was continued horizontally (Figures 5 and 6). The scaled velocities were taken as 3950 m/sec for water, 6000 m/sec for Plexiglas, and 2650 m/sec for the RTV (Figure 6).

The migrated section with variable velocity is shown in Figure 7. In this figure, the Plexiglas base C under the RTV wedge is defined in the correct location. However, part of the base is missing. This can be attributed to the fact that energy which propagates upward from this portion of the base encounters the steeply dipping side of the wedge at an angle beyond the critical angle for RTV and water. This type of propagation is not accounted for by the exploding reflector model on which the migration is based.

It is interesting to note that the migrated results are extremely sensitive to small changes in RTV velocity. In particular the Plexiglas base under the RTV wedge becomes misaligned whenever this velocity is perturbed, in the same manner as the misalignment in Figure 5. In fact, the RTV velocity of 2650 m/sec used for the migration is about 300 m/sec higher than the velocity which is usually quoted for this material. This sensitivity may suggest using migration as a means to determine velocities of physical modeling materials.

CONCLUSIONS

A migration scheme based on a direct integration in depth of the acoustic wave equation has been presented. After the problems of specification of surface boundary conditions and the removal of evanescent energy had been addressed, implementing this migration algorithm was not more complicated than implementing one-way equation schemes.

The present method may offer improvements over conventional finite-difference schemes with regards to the migration of steeply dipping structures, or migration in regions with high velocity contrasts. This is because the full acoustic wave equation is not based on any assumptions concerning the medium through which the waves propagate. Moreover, the numerical algorithm which was used is highly accurate because it uses the Fourier method for calculating horizontal derivatives and a fourth-order Runge-Kutta method for the depth extrapolation. Consequently, for a horizontally stratified medium it proved to be practically equivalent to the analytic phase-shift method (Gazdag, 1978).

The possibility of using the full acoustic wave equation for migration, which was demonstrated in this study for stacked sections, may gain added significance for migration of nonstacked time sections. There, the preservation of amplitude information becomes important and therefore a migration scheme based on the full acoustic wave equation may be necessary to achieve satisfactory results.

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REFERENCES

Berkhout, A. J., and Van Wulfften Palthe, D. W., 1979, Migration in terms of spatial deconvolution: Geophys. Prosp., v. 27, p. 261–291.
Claerbout, J. F., 1976, Fundamentals of geophysical data processing: New York, McGraw-Hill Book Co., Inc.
Claerbout, J. F., and Doherty, S. M., 1972, Downward continuation of moveout corrected seismograms: Geophysics, v. 37, p. 741–768.
Gazdag, J., 1980, Wave equation migration with the accurate space derivative method: Geophys. Prosp., v. 28, p. 60–70.
Kosloff, D., and Baysal, E., 1982, Forward modeling by a Fourier method: Geophysics, v. 47, p. 1402–1412.
Loewenthal, D., Lu, L., Roberson, R., and Sherwood, J. W. C., 1976, The wave equation applied to migration: Geophys. Prosp., v. 24, p. 380–399.
Larner, K., Hatton, L., Gibson, B., and Hsu, I., 1981, Depth migration of imaged time sections: Geophysics, v. 46, p. 734–750.
Mathews, J., and Walker, R. L., 1964, Mathematical methods of physics: The Benjamin/Cummings Publ. Co.
Stolt, R. H., 1978, Migration by Fourier transform: Geophysics, v. 43, p. 23–48.

APPENDIX
WAVE AMPLITUDES FOR 1-D ONE-WAY WAVE EQUATIONS

This appendix shows that one-way wave equations fail to reproduce correct amplitudes for 1-D propagation.

A wide class of one-way equations are derived from a series expansion of the dispersion relation

$$\frac{\omega}{C} = \mp (K_x^2 + K_z^2)^{1/2} \qquad \text{(A–1)}$$

(e.g., Claerbout, 1976, p. 202), where K_z and K_x are, respectively, the vertical and horizontal wavenumbers, ω is the temporal frequency, and C is the acoustic velocity which is assumed to vary slowly in space. The 15 degree wave equation (Claerbout, 1976), for example, can be derived by retaining the first two terms in a Taylor series expansion of equation (A–1) and by replacing the dispersion relation by a differential equation and transforming the result to the moving coordinate system of Claerbout (1976).

In 1-D vertical propagation, K_x in equation (A–1) is set to zero and the one-way wave equation which corresponds to equation (A–1) becomes

$$\frac{1}{C(z)} \frac{\partial P}{\partial t} = \mp \frac{\partial P}{\partial z}. \qquad \text{(A–2)}$$

Equation (A–2) can also be derived directly from most one-way wave equations by setting all terms containing horizontal derivatives to zero. Equation (A–2) can be solved by using the variable separation

$$P = e^{i\omega t} U(z).$$

We obtain

$$P(z,t) = \exp\left[\mp i\omega\left(t - \int_0^z \frac{dy}{C(y)}\right)\right]. \qquad \text{(A–3)}$$

Equation (A–3) is the zeroth order WKB solution to the full acoustic wave equation (Mathews and Walker, 1964). With this solution the amplitude is constant throughout the medium regardless of the values of $C(z)$. Conversely, it may be recalled that the first-order WKB solution to the acoustic wave equation is given by

$$P(z,t) = \frac{1}{(\omega/C)^{1/4}} \exp\left[\mp i\omega\left(t - \int_0^z \frac{dy}{C(y)}\right)\right] \qquad \text{(A–4)}$$

(Mathews and Walker, 1964). This solution does contain amplitude modulation depending upon $C(z)$.

In the case of velocity jumps at interfaces, the one-way wave equation solution (A–3) gives the same wave amplitudes at both sides of the interfaces. On the other hand, for the acoustic wave equation, these amplitudes differ by the ratio of the transmission coefficients.

In conclusion, for both smooth and discontinuous velocity variation, the one-way wave equation does not produce the correct amplitude. These results also apply to wave propagation in two or three spatial dimensions.